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Integral Representation of a Finite Spike

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Consider the function f defined by the following integral:

$$f(u) = \int_{-\pi}^{\pi} \frac{(u + \cos \alpha)(1 + u \cos \alpha)}{(1 + u^2 + 2u \cos \alpha)^{3/2}} \, d\alpha, \tag{1}$$

where $u \geq 0$. We will show that

$$f(u) = \begin{cases} 2, & u = 1 \\ 0, & u \neq 1. \end{cases} \tag{2}$$

Thus the innocent-looking integral in (1) represents a discontinuous function which is zero everywhere except for a finite spike at the point $u = 1$ (FIGURE 1). We ran across this integral in the course of a stability calculation for the equilibrium states of a certain many-body dynamical system.

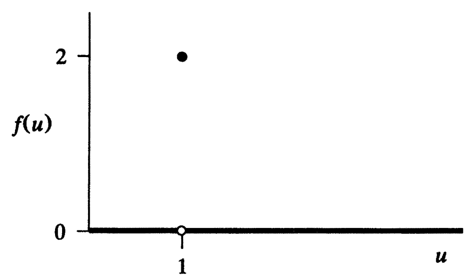

FIG. 1

The most straightforward proof of (2) involves three steps: (a) Show $f(1/u) = uf(u)$. Thus it suffices to prove (2) for $u \leq 1$. (b) Verify $f(u) = 0$ for $0 \leq u < 1$ by a series expansion in powers of u. (c) Compute $f(1) = 2$ as an elementary definite integral.

There is a more appealing proof of (2). It involves the fact that the integrand in (1) may be expressed explicitly as an exact differential. To see this, rewrite the numerator in (1) as $u + u\cos^2 \alpha + u^2 \cos \alpha + \cos \alpha = u \sin^2 \alpha + \cos \alpha(1 + u^2 +$

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$2u\cos\alpha$), using the identity $u = u\sin^2\alpha + u\cos^2\alpha$. Thus (1) becomes

$$f(u) = \int_{-\pi}^{\pi}\left[\frac{u\sin^2\alpha}{(1+u^2+2u\cos\alpha)^{3/2}} + \frac{\cos\alpha}{(1+u^2+2u\cos\alpha)^{1/2}}\right]d\alpha$$

$$= \int_{-\pi}^{\pi}\frac{d}{d\alpha}\left[y_u(\alpha)\right]d\alpha$$

$$= y_u(\pi) - y_u(-\pi), \tag{3}$$

where

$$y_u(\alpha) = \frac{\sin\alpha}{(1+u^2+2u\cos\alpha)^{1/2}}. \tag{4}$$

When $u \neq 1$, $y_u(\pi) = y_u(-\pi) = 0$, and hence (3) implies that $f(u) = 0$. Now assume $u = 1$. Then $(1 + u^2 + 2u\cos\alpha)^{1/2} = \sqrt{2}(1+\cos\alpha)^{1/2} = 2\cos(\alpha/2)$, and so $y_1(\alpha) = (\sin\alpha)/(2\cos(\alpha/2)) = \sin(\alpha/2)$ if $\alpha \neq \pm\pi$. Hence $\lim_{\alpha\to\pm\pi} y_1(\alpha) = \pm 1$. Thus (3) implies $f(1) = 1 - (-1) = 2$, as required.

There is a simple geometric interpretation of these results (Figure 2). Consider two circles of unit radius, and let u be the distance between their centers. Choose a coordinate system with the origin at the center of one of the circles, and the x-axis along the line joining the centers. Given the point $P = (u + \cos\alpha, \sin\alpha)$, define the point Q by the projection shown in FIGURE 2.

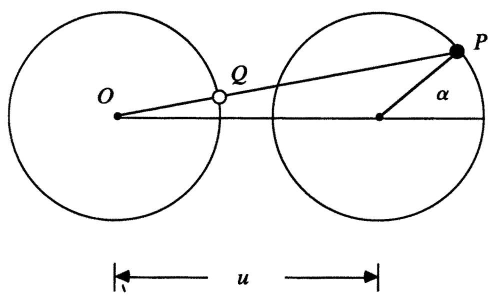

FIG. 2

Then the function $y_u(\alpha)$ may be interpreted as the y-coordinate of Q, since by similar triangles $y(Q) = y(Q)/\|OQ\| = y(P)/\|OP\| = \sin\alpha/(1 + u^2 + 2u\cos\alpha)^{1/2} = y_u(\alpha)$. For $u \neq 1$, Q executes a closed path as α runs from $-\pi$ to π. Thus the net change in $y_u(\alpha)$ along the path is zero, which shows $f(u) = 0$ when $u \neq 1$. However, note that for u near 1, the points P, P' are splayed apart by the mapping $P \mapsto Q$ (FIGURE 3). In fact, for $u = 1$, Q is not defined when

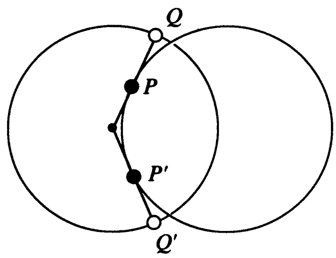

FIG. 3

$\alpha = \pm\pi$. As $\alpha \to \pm\pi$, $Q \to (0, \pm 1)$ and so $y_u(\alpha)$ changes by 2 as α runs from $-\pi$ to π. Thus $f(1) = 2$. Note that the mapping $P \mapsto Q$ cannot be extended to a continuous map at $u = 1$, $\alpha = \pm\pi$. It is this discontinuity which underlies the curious result (2).

On the Irrationality of π^2

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1. Introduction. In 1761 Lambert [**7**] showed, using continued fractions, that $\tan\alpha$ is not rational when α is a nonzero rational from which it follows immediately that π is irrational. Both these results were new; the latter being but a prelude to Lindemann's celebrated proof of 1882 that π is transcendental.

That in fact π^2 is irrational by much the same reasoning with continued fractions was certainly known to Legendre [**8**, Note IV] by 1808. In 1873 Hermite [**2**] showed the irrationality of π^2 essentially from the nontrivial observation that, for the positive integer n,

$$\frac{1}{n!}\left(\frac{\pi}{2}\right)^{2n+1}\int_{-1}^{1}(1-x^2)^n\cos\frac{\pi}{2}x\,dx$$

is a polynomial in $\pi^2/4$ with integer coefficients and degree the integer part of $n/2$. It is then easy to deduce, on the assumption that π^2 is rational, the untenable existence of a positive integer less than 1. Variations on this theme are much favoured in text-books discussing the irrationality of π. Furthermore Hermite's method has been taken up and extended by others; most generally by Inkeri and Estermann. In [**5**] (but see also [**4**] for a similar, though earlier, version in English) Inkeri shows that there is no complex number α with α^2 and $\alpha^{-1}\tan\alpha$ both rational (except possibly 0 if one allows $\alpha^{-1}\tan\alpha = 1$ when $\alpha = 0$ on the grounds of continuity). Later Estermann [**1**], unaware of Inkeri's result, proved an equivalent form very elegantly.

The purpose here is to show yet another path through this well-trodden terrain. Specifically we offer a novel proof of the following theorem which subsumes Inkeri's result.